The hodograph method applicability in the problem of long-scale nonlinear dynamics of a thin vortex filament near a flat boundary

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Hamiltonian dynamics of a thin vortex filament in ideal incompressible fluid near a flat fixed boundary is considered at the conditions that at any point of the curve determining shape of the filament the angle between tangent vector and the boundary plane is small, also the distance from a point on the curve to the plane is small in comparison with the curvature radius. The dynamics is shown to be effectively described by a nonlinear system of two (1+1)-dimensional partial differential equations. The hodograph transformation reduces that system to a single linear differential equation of the second order with separable variables. Simple solutions of the linear equation are investigated at real values of spectral parameter λ when the filament projection on the boundary plane has shape of a two-branch spiral or a smoothed angle, depending on the sign of λ.

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I. INTRODUCTION

It is a well known fact that solutions of equations determining the motion of a homogeneous inviscid fluid possess a remarkable property — the lines of the vorticity field are frozen-in [1, 2, 3, 4]. Mathematical reason for this is the so called relabeling symmetry of fluids that provides necessary conditions for the Noether theorem applicability and results in infinite number of the conservation laws [5, 6, 7, 8, 9, 10, 11] . Due to this basic property, in the framework of ideal hydrodynamics such flows are possible where during sufficiently long time interval the vorticity is concentrated in quasi-one-dimensional structures, vortex filaments, that fill a small part of entire bulk of the fluid. The motion of vortex filaments is very interesting problem both from theoretical and practical viewpoints and is a classical subject of hydrodynamics (see, for instance, [3, 4, 10, 11, 12, 13, 14, 15, 16, 17] and references therein for various analytical and numerical approaches to this problem). In general case analytical study in this field is highly complicated because of several reasons, the main of them being non-locality and nonlinearity of governing equations of motion. A less significant trouble seems to be the necessity of some regularization procedure for the Hamiltonian functional (the total energy) of the system in the limit of “infinitely thin” vortex filaments, since a logarithmic divergency takes place in some observable physical quantities (for instance, in the velocity of displacement of curved pieces of the filament) as the thickness decreases. However, in few limit cases the dynamics of a single vortex filament can turn out to be effectively integrable. A known and very interesting example of such integrable system is a slender non-stretched vortex filament (in the boundless three-dimensional (3D) space filled by an ideal fluid) in the so called localized induction approximation (LIA), when in the energy of the filament only logarithmically large contributions from interaction of adjacent pieces are taken into account. In this approximation the Hamiltonian is simply proportional to the length of the filament, resulting in conservation this quantity, thus application of the so called Hasimoto transformation [18, 19] is appropriate and reduces the problem to (1+1)-dimensional nonlinear Schroedinger equation that is known to be integrable by the inverse scattering method [20].

In present work another integrable case in vortex dynamics is recognized, the long-scale motion of a thin vortex filament near a flat fixed boundary. Mathematically the problem of a single filament in a half-space is equivalent to the problem of two symmetric filaments in the boundless space, that allows us simplify some further calculations. Our immediate purpose will be to consider the configurations of the vortex filament that satisfy the following conditions:

a) the angle is everywhere small between the tangent vector on the curve determining the shape of the filament and the boundary plane;

b) the distance from an arbitrary point of the curve to the plane is small comparatively to the curvature radius at the given point but large in comparison with the thickness of the filament;

c) the filament projection on the boundary plane does not have self-intersections or closely approaching one-to-another different pieces.

In these conditions the system dynamics is known to be unstable (the so called Crow instability [21]), with the instability increment directly proportional to the wave number of (some small) long-scale perturbation of the filament shape. It is a well known fact that such dependence of the increment is usual for a class of local (2×2) partial differential systems that can be exactly linearized by so called hodograph transformation [1] exchanging dependent and independent variables. This observation has served as a weighty reason to look for a natural local nonlinear approximation in description of the long-scale dynamics of a vortex filament near a flat boundary and to

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examine that approximation for the hodograph method applicability. As the result, a consistent derivation of the corresponding local approximate equations of motion has been performed, and also the fact has been demonstrated that the nonlinear partial differential system for two functions ρ and ϑ determining the shape of the filament and depending on the time moment t and on the Cartesian coordinate x is reduced by hodograph transformation to a linear equation. Moreover, it is possible to choose the pair of new independent variables in such a manner that in the linear partial differential equation for the function $t(\rho, \vartheta)$ the coefficients will not depend on ϑ-variable. For this purpose it is convenient to define ρ-variable as double distance from a filament point to the boundary plane $y = 0$, while ϑ-variable will be the angle between x-direction and the tangent to the filament projection on (x, z)-plane. Obviously, an explicit dependence of the coefficients on ϑ will be absent due to the symmetry of the system with respect to rotations in (x, z)-plane. Therefore separation of the variables will be possible and most simple solutions will have the form

$$t_\lambda(\rho, \vartheta) = \mathrm{Re}\left\{T_\lambda(\rho)\Theta_\lambda(\vartheta)\right\}, \tag{1}$$

where λ is an arbitrary complex parameter, and the complex function $\Theta_\lambda(\vartheta)$ satisfies the simple equation

$$\Theta''_\lambda(\vartheta) = \lambda\Theta_\lambda(\vartheta). \tag{2}$$

To find the complex function $T_\lambda(\rho)$, it will be necessary to solve some ordinary linear differential equation of the second order with variable coefficients that will be considered in appropriate section of this paper. The corresponding geometrical configurations of the vortex filament strongly depend on λ. In particular, it will be shown the solutions (1) with real $\lambda < -1$ describe such a shape of the (moving) vortex filament that its projection on (x, z)-plane has two asymptotes with the angle between them $\Delta\vartheta = \pi(1 - 1/\sqrt{-\lambda})$, while in the case $\lambda > 0$ the projection has the shape of a two-branch spiral (see the figures).

This article is organized as follows. In section II a necessary brief review is given concerning the Hamiltonian formalism adopted to the problem of frozen-in vorticity, since this approach is the most clear and compact way to treat ideal flows. Then approximate local equations of motion for a vortex filament near a flat boundary are derived. In section III we demonstrate applicability of the hodograph method and introduce variables that are most convenient for the particular problem. Section IV is devoted to investigation of simple solutions obtained by the variables separation in the governing linear equation.

II. LONG-SCALE LOCAL APPROXIMATION

Existence itself of the ideal-hydrodynamic solutions in the form of quasi-one-dimensional vortex structures (vortex filaments) filling just a small part of the total fluid bulk is closely connected with the freezing-in property of the vortex lines [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Mathematically this property is expressed by the special form of the equation of motion for the divergence-free vorticity field $\boldsymbol{\Omega}(\boldsymbol{r}, t) = \mathrm{curl}\, \boldsymbol{v}(\boldsymbol{r}, t)$,

$$\boldsymbol{\Omega}_t = \mathrm{curl}\,[\boldsymbol{v} \times \boldsymbol{\Omega}], \tag{3}$$

where $\boldsymbol{v}(\boldsymbol{r}, t)$ is the velocity field. Since in this article we consider incompressible flows, we may write

$$\boldsymbol{v} = \mathrm{curl}^{-1}\boldsymbol{\Omega} = \mathrm{curl}\,(-\Delta)^{-1}\boldsymbol{\Omega}, \tag{4}$$

where Δ is the 3D Laplace operator. As it is known, the action of the inverse nonlocal operator Δ^{-1} on an arbitrary function $f(\boldsymbol{r})$ is given by the formula

$$-\Delta^{-1}f(\boldsymbol{r}) = \int G(|\boldsymbol{r} - \boldsymbol{r}_1|)f(\boldsymbol{r}_1)d\boldsymbol{r}_1, \tag{5}$$

where

$$G(r) = \frac{1}{4\pi r}$$

is the Green function of the $(-\Delta)$ operator in the boundless space. Hamiltonian noncanonical structure [7] of the equations of ideal incompressible hydrodynamics is based on the relation

$$\boldsymbol{v} = \mathrm{curl}\left(\frac{\delta\mathcal{H}}{\delta\boldsymbol{\Omega}}\right), \tag{6}$$

where the Hamiltonian functional $\mathcal{H}\{\boldsymbol{\Omega}\}$ is the kinetic energy of a homogeneous incompressible fluid (with unit density) expressed through the vorticity,

$$\mathcal{H}\{\boldsymbol{\Omega}\} = \frac{1}{2}\int \boldsymbol{\Omega}\cdot(-\Delta)^{-1}\boldsymbol{\Omega}\, d\boldsymbol{r}. \tag{7}$$

Our approach to investigation of the vortex filament motion is based on representation of the ideal homogeneous fluid flows in terms of the frozen-in vortex lines, as described, for instance, in [10, 11, 12, 13]. The special form (3) of the equation of motion allows us express the vorticity field $\boldsymbol{\Omega}(\boldsymbol{r}, t)$ in a self-consistent manner through the shape of the vortex lines (the so called formalism of vortex lines),

$$\boldsymbol{\Omega}(\boldsymbol{r}, t) = \int_{\mathcal{N}} d^2\nu \oint \delta(\boldsymbol{r} - \boldsymbol{R}(\nu, \xi, t))\boldsymbol{R}_\xi(\nu, \xi, t)d\xi, \tag{8}$$

where $\delta(\ldots)$ is the 3D delta-function, $\mathcal{N}$ is some 2D manifold of labels enumerating the vortex lines ($\mathcal{N}$ is determined by topological properties of a particular flow), $\nu \in \mathcal{N}$ is a label of an individual vortex line, ξ is an arbitrary longitudinal parameter along the line. What is important, the dynamics of the line shape $\boldsymbol{R}(\nu, \xi, t) = (X(\nu, \xi, t), Y(\nu, \xi, t), Z(\nu, \xi, t))$ is determined by the variational principle

$$\delta\left[\int \mathcal{L}dt\right]/\delta\boldsymbol{R}(\nu, \xi, t) = 0,$$

with the Lagrangian of the form

$$\mathcal{L} = \int_{\mathcal{N}} d^2\nu \oint ([\boldsymbol{R}_\xi \times \boldsymbol{R}_t] \cdot \boldsymbol{D}(\boldsymbol{R})) d\xi \; - \mathcal{H}\{\boldsymbol{\Omega}\{\boldsymbol{R}\}\}, \quad (9)$$

where the vector function $\boldsymbol{D}(\boldsymbol{R})$ in the case of incompressible flows must satisfy the condition

$$(\nabla_{\boldsymbol{R}} \cdot \boldsymbol{D}(\boldsymbol{R})) = 1. \quad (10)$$

Below we choose $\boldsymbol{D}(\boldsymbol{R}) = (0, Y, 0)$.

Since we are going to deal with a very thin vortex filament, we will neglect the ν-dependence of the shapes of individual vortex lines constituting the filament. By doing this step we exclude from further consideration all effects related to finite variable cross-section and longitudinal flows inside the filament. Thus, we consider an "infinitely narrow" vortex string with a shape $\boldsymbol{R}(\xi, t)$ and with a finite circulation $\Gamma = \int_{\mathcal{N}} d^2\nu$. However, the Hamiltonian of such singular filament diverges logarithmically,

$$\mathcal{H}_\Gamma\{\boldsymbol{R}(\xi)\} = \frac{\Gamma^2}{8\pi} \oint\oint \frac{\boldsymbol{R}'(\xi_1) \cdot \boldsymbol{R}'(\xi_2)\; d\xi_1\, d\xi_2}{|\boldsymbol{R}(\xi_1) - \boldsymbol{R}(\xi_2)|} \;\to \infty \quad (11)$$

In order to regularize this double integral, it is possible, as a variant, to modify the Green function [13]. For example, instead of the singular function $G \propto 1/r$ one can use a smooth function like $G_a \propto 1/\sqrt{r^2 + a^2}$ or some other appropriate expression depending on a parameter a. It should be emphasized that relation $\boldsymbol{\Omega} = \text{curl}\, \boldsymbol{v}$ is exactly satisfied only in the original non-regularized system, but in the case of a finite a it is not valid on distances of order a from the singular vortex string. Thus, the meaning of vorticity in regularized models is not so simple, but nevertheless, relation (6) remains valid in any case. Relatively small parameter a in regularized models serves to imitate a finite width of vortex filament in the usual (non-regularized) hydrodynamics. The energy of the string turns out to be logarithmically large,

$$\mathcal{H}_\Gamma\{\boldsymbol{R}(\xi)\} \approx \frac{\Gamma^2}{4\pi} \oint |\boldsymbol{R}'(\xi)| \log\left(\frac{\Lambda(\boldsymbol{R}(\xi))}{a}\right) d\xi, \quad (12)$$

where $\Lambda(\boldsymbol{R})$ is a typical scale depending on a given problem (in particular, the usual LIA uses $\Lambda = \text{const} \gg a$). In our case we consider two symmetric vortex filaments in the long-scale limit, when direction of the tangent vector varies weakly on a length of order Y. For such configurations, the energy concentrated in the half-space $y > 0$ is approximately equal to the following expression

$$\mathcal{H}_\Gamma \approx \frac{\Gamma^2}{4\pi} \oint \sqrt{X'^2 + Y'^2 + Z'^2}\, \log(2Y/a) d\xi. \quad (13)$$

This local Hamiltonian is able to provide qualitatively correct dynamics of the filament down to longitudinal scales of order Y where perturbations become stable and where non-locality comes to play. Unfortunately, we do not have a simple method to treat the Hamiltonian (13) analytically, that is why we will consider only very large scales ($L \gg Y$) and thus suppose the slope of the tangent vector to the boundary plane to be negligibly small (this means $Y'^2 \ll X'^2 + Z'^2$). Then, choosing as a longitudinal parameter ξ simply the Cartesian coordinate x, we have the following approximate Lagrangian:

$$\mathcal{L} \approx \Gamma \int \left\{ -Y\dot{Z} - \frac{\Gamma}{4\pi}\sqrt{1 + Z'^2}\log(2Y/a) \right\} dx, \quad (14)$$

where the functions Y and Z depend on x and t, $\dot{Z} \equiv \partial_t Z$, $Z' \equiv \partial_x Z$. Having neglected the term Y'^2 under the square root, we sacrifice correct behaviour of perturbations with wave-lengths of order Y, but instead we obtain exactly solvable system, as it will be shown below.

Let us say a few words about geometrical meaning of the second term in r.h.s. of the expression (14). Since we study the very long-scale limit, locally the flow under consideration looks almost like a two-dimensional flow with a small vortex at the distance Y from the straight boundary, and the expression $(\Gamma/4\pi)\log(2Y/a)$ is just the energy of such 2D flow per unit length in the third (longitudinal) direction, while the multiplier $\sqrt{1 + Z'^2}\, dx$ gives the arc-length element in the longitudinal direction.

Now for simplicity we take new time and length scales to satisfy $a = 1$ and $\Gamma/2\pi = 1$. After that we introduce new quantities $\rho(x,t) = 2Y(x,t)$ and $\mu(x,t) = \partial_x Z(x,t)$, and also the function $H(\rho, \mu)$,

$$H(\rho, \mu) = F(\rho)\sqrt{1 + \mu^2}, \quad (15)$$

where

$$F(\rho) = \log \rho. \quad (16)$$

The corresponding equations of motion then can be written in the following remarkable general form:

$$\mu = \partial_x Z, \quad (17)$$

$$\partial_t \rho + \partial_x H_\mu(\rho, \mu) = 0, \quad (18)$$

$$\partial_t Z + H_\rho(\rho, \mu) = 0. \quad (19)$$

More explicitly, the last two equations are

$$\rho_t + \frac{\partial}{\partial x}\left[\frac{F(\rho) Z_x}{\sqrt{1 + Z_x^2}}\right] = 0, \quad (20)$$

$$\partial_t Z + F'(\rho)\sqrt{1 + Z_x^2} = 0. \quad (21)$$

These equations have a simple geometrical treatment. Indeed, Eq.(21) means if we consider the dynamics of the filament projection on (x, z)-plane, then we see an element of the projection moving in the normal to the projection tangent direction with the velocity depending only on ρ and equal to $F'(\rho)$. Simultaneously, in y-direction the element of the filament moves with the velocity proportional to the (x, z)-projection curvature multiplied by the function $F(\rho)$, as Eq.(20) shows.

It is interesting to note that an analogous consideration can give us also long-scale Hamiltonian equations of motion for a thin vortex filament in a slab of an ideal fluid between two parallel fixed boundaries at $y = -d/2$ and $y = +d/2$. One has just to define the ρ-variable by the formula $\rho = (\pi/d)y$ and make in Eq.(15) substitution $F(\rho) \mapsto F^{(\epsilon)}(\rho)$ where

$$F^{(\epsilon)}(\rho) = \log\left(\frac{\cos\rho}{\epsilon}\right), \tag{22}$$

with a small dimensionless parameter ϵ.

III. HODOGRAPH METHOD

It is known that any nonlinear system of the form (17)-(19) can be locally reduced to a linear equation if we take ρ and μ as new independent variables (this is the so called hodograph method; see, for instance, [1] where a particular case is discussed, the 1D gas dynamics, with $H(\rho,\mu) = \rho\mu^2/2 + \varepsilon(\rho)$, where ρ, μ, $\varepsilon(\rho)$ are the gas density, gas velocity, and the internal energy density respectively). Indeed, as from equations (17) and (19) we see the relation

$$dZ = \mu\, dx - H_\rho\, dt,$$

it is convenient to introduce the auxiliary function $\chi(\rho,\mu)$,

$$\chi = Z - x\mu + tH_\rho \tag{23}$$

in order to obtain

$$d\chi = -x\, d\mu + tH_{\rho\rho}\, d\rho + tH_{\rho\mu}\, d\mu.$$

From the above expression we easily derive

$$t = \frac{\chi_\rho}{H_{\rho\rho}}, \qquad x = H_{\rho\mu}\frac{\chi_\rho}{H_{\rho\rho}} - \chi_\mu. \tag{24}$$

After that we rewrite Eq.(18) in the form

$$\frac{\partial(\rho,x)}{\partial(t,x)} - H_{\mu\rho}\frac{\partial(\rho,t)}{\partial(t,x)} - H_{\mu\mu}\frac{\partial(\mu,t)}{\partial(t,x)} = 0$$

and multiply it by the Jacobian $\partial(t,x)/\partial(\rho,\mu)$,

$$\frac{\partial(\rho,x)}{\partial(\rho,\mu)} - H_{\mu\rho}\frac{\partial(\rho,t)}{\partial(\rho,\mu)} - H_{\mu\mu}\frac{\partial(\mu,t)}{\partial(\rho,\mu)} = 0.$$

Thus, now we have

$$x_\mu = H_{\mu\rho}t_\mu - H_{\mu\mu}t_\rho. \tag{25}$$

Substitution of the relations (24) into this equation and subsequent simplification give us the linear partial differential equation of the second order for the function $\chi(\rho,\mu)$,

$$(H_{\mu\mu}\chi_\rho/H_{\rho\rho})_\rho - \chi_{\mu\mu} = 0. \tag{26}$$

As the function $H(\rho,\mu)$ has the special form (15), it is convenient to change variables,

$$\mu = \tan\vartheta, \qquad \chi(\rho,\vartheta) = -\frac{u(\rho,\vartheta)}{\cos\vartheta}, \tag{27}$$

where ϑ is the angle in (x,z)-plane between x-direction and the tangent to the corresponding projection of the vortex filament. As the result, the relations (23) and (24) will be rewritten in the form

$$t = -\frac{u_\rho}{F''(\rho)} \tag{28}$$

$$x = u_\vartheta\cos\vartheta + \left(u - \frac{F'(\rho)}{F''(\rho)}u_\rho\right)\sin\vartheta, \tag{29}$$

$$Z = u_\vartheta\sin\vartheta - \left(u - \frac{F'(\rho)}{F''(\rho)}u_\rho\right)\cos\vartheta, \tag{30}$$

and the coefficients of the linear equation for the function $u(\rho,\vartheta)$ will not depend on ϑ-variable,

$$\frac{\partial}{\partial\rho}\left(\frac{F(\rho)}{F''(\rho)}u_\rho\right) - (u_{\vartheta\vartheta} + u) = 0. \tag{31}$$

The same is true for the coefficients of the equation determining the function $t(\rho,\vartheta) = -u_\rho(\rho,\vartheta)/F''(\rho)$,

$$F(\rho)t_{\rho\rho} + 2F'(\rho)t_\rho - F''(\rho)t_{\vartheta\vartheta} = 0. \tag{32}$$

Once some particular solution of Eq.(31) is known, then further procedure consists in the following two steps:

i) find in terms of some parameter ξ the curves of constant values of the function $t = -u_\rho(\rho,\vartheta)/F''(\rho)$. It is this point where nonlinearity comes to play, since we need to solve nonlinear equation;

ii) substitute the obtained expressions $\rho = \rho(\xi,t)$ and $\vartheta = \vartheta(\xi,t)$ into Eqs.(29-30) and get complete description of the filament motion, $X = X(\xi,t)$, $Z = Z(\xi,t)$, $Y = (1/2)\rho(\xi,t)$.

Thus, the long-scale local approximation (14) turns out to be integrable in the sense it is reduced to the *linear* equation (31). However, the function $u(\rho,\vartheta)$ is multi-valued in the general case. Therefore statement of the Cauchy problem becomes much more complicated. Besides, the functions $F(\rho)$ and $F^{(\epsilon)}(\rho)$ determined by expressions (16) and (22) result in *elliptic* linear equations as against the usual 1D gas dynamics where the corresponding equations were *hyperbolic*. Generally speaking, the ellipticity makes the Cauchy problem ill-posed in the mathematical sense if initial data are not very smooth. However, in this article we will not discuss these questions, instead in the following section we will present simple particular solutions that during some time interval satisfy the applicability conditions for the long-scale approximation.

IV. PARTICULAR SOLUTIONS

A. Separation of the variables

We are going to consider the simplest particular solutions of Eq.(31) obtainable by separation of the variables

$$u_\lambda(\rho,\vartheta) = \mathrm{Re}\left\{U_\lambda(\rho)\Theta_\lambda(\vartheta)\right\}, \tag{33}$$

where λ is an arbitrary complex spectral parameter,

$$\lambda = (\varkappa + ik)^2, \qquad k \geq 0, \tag{34}$$

and the function $\Theta_\lambda(\vartheta)$ contains two arbitrary complex constants C^+_λ and C^-_λ,

$$\Theta_\lambda(\vartheta) = C^+_\lambda \exp[(\varkappa+ik)\vartheta] + C^-_\lambda \exp[-(\varkappa+ik)\vartheta]. \tag{35}$$

The motion of the vortex filament will be described by the formulas

$$t_\lambda = -\mathrm{Re}\left\{\frac{U'_\lambda(\rho)}{F''(\rho)}\Theta_\lambda(\vartheta)\right\}, \tag{36}$$

$$x_\lambda = \mathrm{Re}\Big\{U_\lambda(\rho)\Theta'_\lambda(\vartheta)\cos\vartheta + \Big(U_\lambda(\rho) - \frac{F'(\rho)}{F''(\rho)}U'_\lambda(\rho)\Big)\Theta_\lambda(\vartheta)\sin\vartheta\Big\}, \tag{37}$$

$$Z_\lambda = \mathrm{Re}\Big\{U_\lambda(\rho)\Theta'_\lambda(\vartheta)\sin\vartheta - \Big(U_\lambda(\rho) - \frac{F'(\rho)}{F''(\rho)}U'_\lambda(\rho)\Big)\Theta_\lambda(\vartheta)\cos\vartheta\Big\}. \tag{38}$$

The function $U_\lambda(\rho)$ must satisfy the ordinary differential equation of the second order

$$\frac{d}{d\rho}\left(\frac{F(\rho)}{F''(\rho)}U'_\lambda(\rho)\right) - (\lambda+1)U_\lambda(\rho) = 0. \tag{39}$$

Let us turn a bit of attention to the special value $\lambda = -1$ of the spectral parameter, when the solution of Eq.(39) can be explicitly written for any function $F(\rho)$,

$$U_{-1}(\rho) = A_{-1}\int^\rho \frac{F''(\rho_1)d\rho_1}{F(\rho_1)} + B_{-1}, \tag{40}$$

where A_{-1} and B_{-1} are arbitrary complex constants.

At $\lambda \neq -1$ it can be convenient to deal with the function

$$T_\lambda(\rho) = -\frac{U'_\lambda(\rho)}{F''(\rho)}, \tag{41}$$

that satisfies the equation

$$F(\rho)T''_\lambda(\rho) + 2F'(\rho)T'_\lambda(\rho) - \lambda F''(\rho)T_\lambda(\rho) = 0. \tag{42}$$

In particular, Eq.(42) is simply solvable at $\lambda = 0$ (this solution describes the motion of a perfect vortex ring),

$$T_0(\rho) = A_0\int^\rho \frac{d\rho_1}{F^2(\rho_1)} + B_0. \tag{43}$$

Simple manipulations with formulas (36-39) allow us rewrite the solutions in the form

$$t_\lambda = \mathrm{Re}\Big\{T_\lambda(\rho)\Theta_\lambda(\vartheta)\Big\}, \tag{44}$$

$$x_\lambda = \mathrm{Re}\Big\{(\lambda+1)^{-1}\{-[F(\rho)T_\lambda(\rho)]'\Theta'_\lambda(\vartheta)\cos\vartheta + [\lambda F'(\rho)T_\lambda(\rho) - F(\rho)T'_\lambda(\rho)]\Theta_\lambda(\vartheta)\sin\vartheta\}\Big\}, \tag{45}$$

$$Z_\lambda = \mathrm{Re}\Big\{(\lambda+1)^{-1}\{-[F(\rho)T_\lambda(\rho)]'\Theta'_\lambda(\vartheta)\sin\vartheta - [\lambda F'(\rho)T_\lambda(\rho) - F(\rho)T'_\lambda(\rho)]\Theta_\lambda(\vartheta)\cos\vartheta\}\Big\}. \tag{46}$$

B. Real λ

Let us first consider real values of the spectral parameter, $\lambda \in \mathcal{R}$, and the corresponding real functions $\Theta_\lambda(\vartheta)$ and $U_\lambda(\rho)$. Since $F(\rho) > 0$, $F''(\rho) < 0$, we may expect the solutions $U_\lambda(\rho)$ with $\lambda \gg 1$ to have a number of oscillations, the more, the more λ is. In the opposite case, when $\lambda < -1$, the solutions will be a linear combination of two functions, one of them being increasing, and other decreasing. It is sufficient to know these general properties to get an impression concerning geometrical configurations of the vortex filament described by the formulas (44-46). Let us take $\lambda = -k^2$ with $k > 1$ and suppose the explicit dependence $T_{-k^2}(\rho)$ to be known and increasing at large ρ. For simplicity we take $\Theta_{-k^2}(\vartheta) = \cos(k\vartheta)$ and after that resolve the relation (44) with respect to ϑ,

$$\vartheta = \pm\frac{1}{k}\arccos\left[\frac{t}{T_{-k^2}(\rho)}\right]. \tag{47}$$

Substitution of this expression into formulas (45-46) gives us final form of the solutions as dependences $X_{-k^2}(\rho,t)$ and $Z_{-k^2}(\rho,t)$,

$$X_{-k^2}(\rho,t) = \pm\frac{k^2F'(\rho)T_{-k^2}(\rho) + F(\rho)T'_{-k^2}(\rho)}{k^2-1}\left[\frac{t}{T_{-k^2}(\rho)}\right]\sin\left(\frac{1}{k}\arccos\left[\frac{t}{T_{-k^2}(\rho)}\right]\right) \mp\frac{k[F(\rho)T_{-k^2}(\rho)]'}{k^2-1}\left[1-\frac{t^2}{T^2_{-k^2}(\rho)}\right]^{1/2}\cos\left(\frac{1}{k}\arccos\left[\frac{t}{T_{-k^2}(\rho)}\right]\right), \tag{48}$$

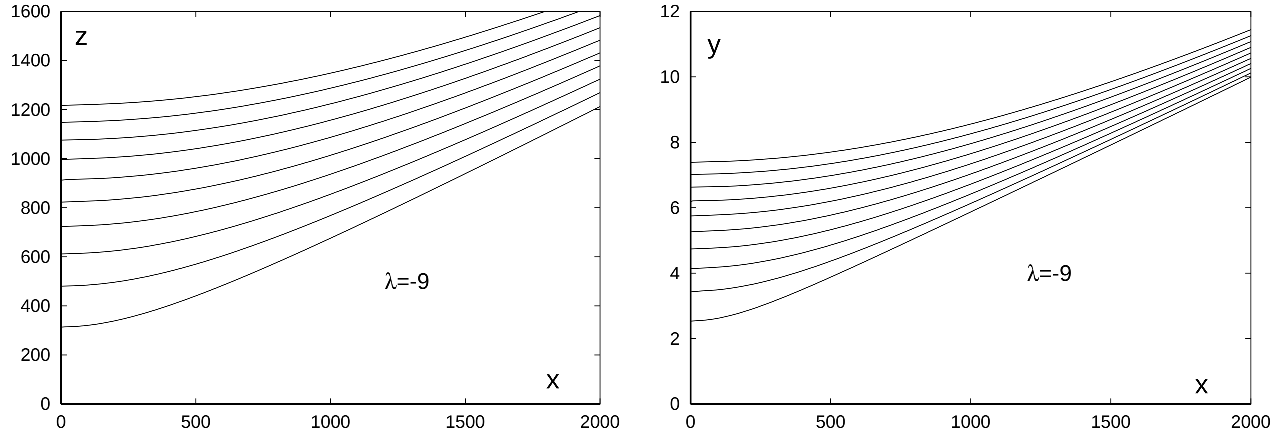


FIG. 1: Solution for $\lambda = -9$.

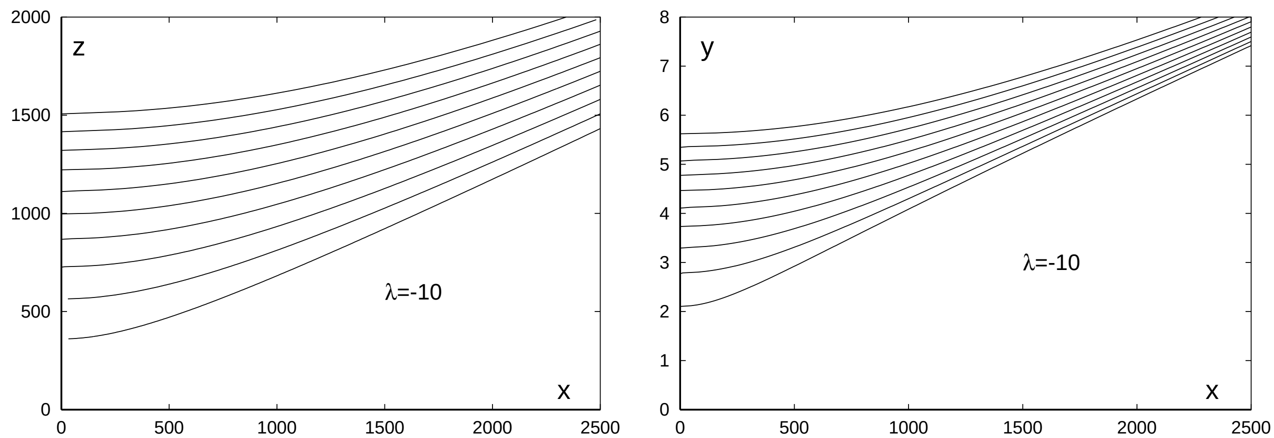


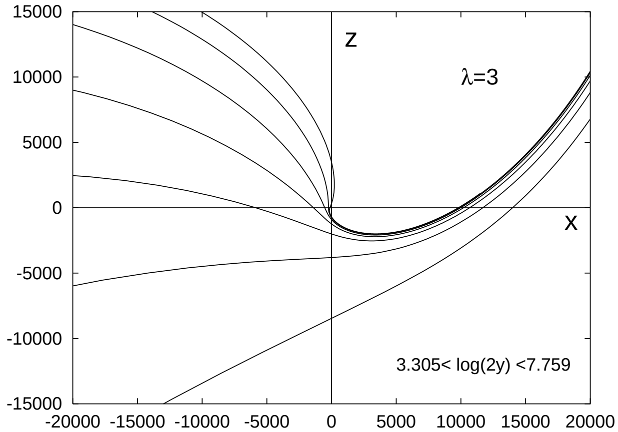
FIG. 2: Solution for $\lambda = -10$.

$$
\begin{aligned}
Z_{-k^2}(\rho,t) \;&=\; \mp\frac{k^2F'(\rho)T_{-k^2}(\rho)+F(\rho)T'_{-k^2}(\rho)}{k^2-1}\left[\frac{t}{T_{-k^2}(\rho)}\right]\cos\left(\frac{1}{k}\arccos\left[\frac{t}{T_{-k^2}(\rho)}\right]\right) \\
&\mp\frac{k[F(\rho)T_{-k^2}(\rho)]'}{k^2-1}\left[1-\frac{t^2}{T^2_{-k^2}(\rho)}\right]^{1/2}\sin\left(\frac{1}{k}\arccos\left[\frac{t}{T_{-k^2}(\rho)}\right]\right).
\end{aligned}
\tag{49}
$$

The ρ-variable in the above expressions varies in the limits from $\rho_{min}(t)$ such that $t = T_{-k^2}(\rho_{min})$, to $+\infty$. The corresponding curve in (x,z)-plane is a smoothed angle $\Delta\vartheta = \pi(1-1/k)$ [see Figs.(1-2) where for the case $F(\rho) = \log\rho$ the filament shape is shown at several time moments, $t_{n+1} - t_n = \text{const}$]. Completely different form is obtained at $\lambda = \varkappa^2$, two-branch spirals [see Fig.(3)]. Let us take $\Theta_{\varkappa^2}(\vartheta) = \exp(\varkappa\vartheta)$. Then

$$
\vartheta = \frac{1}{\varkappa}\log\left[\frac{t}{T_{\varkappa^2}(\rho)}\right],
\tag{50}
$$

$$
\begin{aligned}
X_{\varkappa^2}(\rho,t) \;&=\; \frac{t}{T_{\varkappa^2}(\rho)}\left\{\frac{\varkappa^2F'(\rho)T_{\varkappa^2}(\rho)-F(\rho)T'_{\varkappa^2}(\rho)}{\varkappa^2+1}\sin\left(\frac{1}{\varkappa}\log\left[\frac{t}{T_{\varkappa^2}(\rho)}\right]\right)\right. \\
&\left.-\frac{\varkappa[F(\rho)T_{\varkappa^2}(\rho)]'}{\varkappa^2+1}\cos\left(\frac{1}{\varkappa}\log\left[\frac{t}{T_{\varkappa^2}(\rho)}\right]\right)\right\},
\end{aligned}
\tag{51}
$$



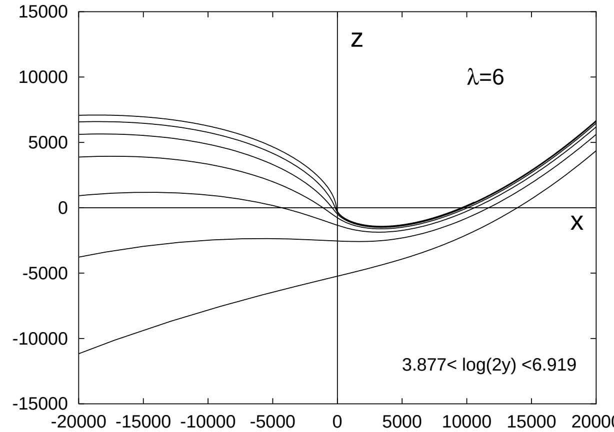


FIG. 3: Two-branch spirals. The filament projection is presented for several time moments, $|t_{n+1}/t_n| = 1/2$.

$$Z_{\varkappa^2}(\rho,t) = \frac{t}{T_{\varkappa^2}(\rho)}\Bigg\{ -\frac{\varkappa^2 F'(\rho)T_{\varkappa^2}(\rho) - F(\rho)T'_{\varkappa^2}(\rho)}{\varkappa^2+1}\cos\left(\frac{1}{\varkappa}\log\left[\frac{t}{T_{\varkappa^2}(\rho)}\right]\right) - \frac{\varkappa[F(\rho)T_{\varkappa^2}(\rho)]'}{\varkappa^2+1}\sin\left(\frac{1}{\varkappa}\log\left[\frac{t}{T_{\varkappa^2}(\rho)}\right]\right)\Bigg\}. \tag{52}$$

The variable ρ runs here between two neighbour zeros of the function $T_{\varkappa^2}(\rho)$ and approaches these values at two logarithmic branches of the spiral, $\rho_j^{(\varkappa)} < \rho < \rho_{j+1}^{(\varkappa)}$.

C. The case $F(\rho) = \log\rho$

For further investigation let us substitute $F(\rho) = \log\rho$ into the equations (36-39) and change the variable

$$q = \log\rho.$$

As the result, we will obtain

$$t_\lambda = \mathrm{Re}\{e^q\, U'_\lambda(q)\Theta_\lambda(\vartheta)\}, \tag{53}$$

$$x_\lambda = \mathrm{Re}\{U_\lambda(q)\Theta'_\lambda(\vartheta)\cos\vartheta + [U_\lambda(q)+U'_\lambda(q)]\,\Theta_\lambda(\vartheta)\sin\vartheta\}, \tag{54}$$

$$Z_\lambda = \mathrm{Re}\{U_\lambda(q)\Theta'_\lambda(\vartheta)\sin\vartheta - [U_\lambda(q)+U'_\lambda(q)]\,\Theta_\lambda(\vartheta)\cos\vartheta\}. \tag{55}$$

$$qU''_\lambda(q) + (1+q)U'_\lambda(q) + (1+\lambda)U_\lambda(q) = 0. \tag{56}$$

General solution of Eq.(56) is representable by the Laplace method [22] as arbitrary linear combination $A_\lambda I^{\mathcal{A}}_\lambda(q) + B_\lambda I^{\mathcal{B}}_\lambda(q)$ of two contour integrals in a complex plane,

$$U_\lambda(q) = \frac{A_\lambda}{2\pi i}\oint_{\mathcal{A}}\left(\frac{p}{p+1}\right)^{\lambda+1}\frac{e^{pq}dp}{p} + B_\lambda\int_{\mathcal{B}}\left(\frac{p}{p+1}\right)^{\lambda+1}\frac{e^{pq}dp}{p}. \tag{57}$$

Here the first closed contour $\mathcal{A}$ goes around the points $p_0 = 0$ and $p_1 = -1$. The second contour $\mathcal{B}$ is not closed, at positive q it starts at $\mathrm{Re}\,p = -\infty$. If $\mathrm{Re}\,\lambda < 0$, then $\mathcal{B}$ ends at p_1, but if $\mathrm{Re}\,\lambda \geq 0$, then its end point is p_0. In both cases at the end point of the contour $\mathcal{B}$ the integrand multiplied by $p(p+1)$ tends to zero.

It is interesting to note that at the integer values of the parameter λ the integral $I^{\mathcal{A}}_\lambda(q)$ can be expressed in terms of polynomials:

$$I^{\mathcal{A}}_\lambda(q) = \frac{1}{2\pi i}\oint_{\mathcal{A}}\left(\frac{p}{p+1}\right)^{\lambda+1}\frac{e^{pq}dp}{p} = \begin{cases} \left(1+\frac{d}{dq}\right)^{|\lambda|-1}\frac{q^{|\lambda|-1}}{(|\lambda|-1)!}, & \lambda = -1,-2,..; \\ e^{-q}\left(\frac{d}{dq}-1\right)^{\lambda}\frac{q^\lambda}{\lambda!}, & \lambda = 0,1,2,... \end{cases} \tag{58}$$

These expressions have been used to prepare Figs.(1)-(3) where the vortex filament shape corresponding to $U_\lambda(q) = I^{\mathcal{A}}_\lambda(q)$ is drawn for several moments of time. It is easily to see, at sufficiently large times the spirals satisfy the conditions a), b), c) that have been formulated in the Introduction. As to the angle-shaped configurations, the condition $Y'^2 \ll X'^2 + Z'^2$, generally speaking, is not satisfied at $q \gtrsim k^2$, since at very large q (on the asymptotes of the angle) the growth of $Y \sim \exp(q)$ is faster than growth of $X, Z \sim U_{-k^2}(q) \sim q^{k^2-1}$. Therefore, if we take a particular solution $u = U_{-k^2}(q)\cos(k\vartheta)$ separately, not as a term in a more complex linear combination, then we have to deal only with large k, and consider only the pieces of the filament where $2..3 \lesssim q \ll k^2$.

D. The case $F(\rho) = \rho^\alpha/\alpha$

In Ref.[13] we investigated another regularization of the Hamiltonian functional that corresponds to $F(\rho) = \rho^\alpha/\alpha$, with some small positive parameter α. That time we did not see applicability of the hodograph method and therefore we were able to find only few particular solutions. Now it has been clear that in this case a simple substitution exists that reduces the problem to 2D equation $\Delta_2 f + f = 0$. Thus, it becomes possible to present a very wide class of solutions of the equation

$$\rho^2 t_{\rho\rho} + 2\alpha\rho\, t_\rho + \alpha(1-\alpha)t_{\vartheta\vartheta} = 0 \tag{59}$$

as linear combinations of singular fundamental solutions (that are expressed through the McDonald function K_0) and regular exponential or polynomial solutions. Indeed, by the substitutions

$$\rho = e^q, \quad \vartheta = \phi\sqrt{\alpha(1-\alpha)}, \quad t = e^{(1/2-\alpha)q} f(q,\phi) \tag{60}$$

Eq.(59) is reduced to the equation with constant coefficients,

$$f_{qq} + f_{\phi\phi} - (1/2-\alpha)^2 f = 0. \tag{61}$$

As it is well known, the fundamental solutions of this equation have the form

$$f(q,\phi;q_0,\phi_0) = K_0\left(\left|\frac{1}{2}-\alpha\right|\sqrt{(q-q_0)^2+(\phi-\phi_0)^2}\right), \tag{62}$$

where q_0 and ϕ_0 are arbitrary parameters. Therefore Eq.(59) has particular solutions

$$t = \rho^{1/2-\alpha} K_0\left(\left|\frac{1}{2}-\alpha\right|\sqrt{\left[\log\frac{\rho}{\rho_0}\right]^2 + \frac{(\vartheta-\vartheta_0)^2}{\alpha(1-\alpha)}}\right). \tag{63}$$

It is interesting to note that at $\alpha = 1/2$ the system possesses conformal symmetry. A deep reason of this symmetry is not clear yet.

As concerning separation of the variables, the function $T_\lambda(\rho)$ in Eqs.(44)-(46) is given by the expression

$$T_\lambda(\rho) = A_\lambda^+ \rho^{s_+(\lambda)} + A_\lambda^- \rho^{s_-(\lambda)}, \tag{64}$$

where $A_\lambda^\pm$ are arbitrary constants. The complex exponents $s_\pm(\lambda)$ are the roots of the quadratic equation

$$s(s-1) + 2\alpha s + \alpha(1-\alpha)\lambda = 0. \tag{65}$$

Thus,

$$s_\pm(\lambda) = 1/2 - \alpha \pm \sqrt{(1/2-\alpha)^2 - \alpha(1-\alpha)\lambda}. \tag{66}$$

It should be mentioned the solutions presented in [13] correspond to the particular case $\alpha + s = 2$.

V. CONCLUSIONS

In this article an approximate exactly solvable nonlinear model has been derived to describe unstable locally-quasi-2D ideal flows with a thin vortex filament near a flat boundary. The hodograph method has been applied and some particular solutions have been analytically found by variables separation in the governing linear partial differential equation for auxiliary function u. More general solutions $u(\rho,\vartheta)$ can be obtained as linear combinations of the terms (33) with different λ, but only in few cases it will be possible to resolve analytically the dependence $t = -u_\rho(\rho,\vartheta)/F''(\rho)$. However, this procedure can be performed numerically.

Though we derived the exactly solvable model under several restrictive simplifications, the solutions obtained in this work promise benefit in many aspects. For instance, they may serve as basic approximations in future more advanced analytical studies that will take into account effects of non-locality and/or finite variable cross-section of the filament, as well as surface waves in the case of free boundary.

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